SB. 2.10.2008

Lie Groups

G - a group which is a manifold s.t. the map

$$G\times G \ni (a,b) \longmapsto ab^{-1} \in G \quad \text{is differentiable}$$

is a Lie Group

Left translations: $\forall a \in G$: $L_a : G \to G$

$L_a(b) = a \cdot b$

Right translations: $R_a : G \to G$

$R_a(b) = b \cdot a$

} are diffeomorphisms

Def A Riemannian metric g on G is left invariant iff

$$g_b(X_b, Y_b) = g_{ab}(L_{a*_b} X_b, L_{a*_b} Y_b)$$

$\forall a,b \in G$
$\forall X_b, Y_b \in T_b G$

(similarly right invariant)

g is biinvariant iff it is left and right invariant.

Def

A vector field X on G is left invariant iff

$$\forall a \in G \quad L_{a*} X = X \qquad (L_{a*_b} X_b = X_{ab} \quad \forall a,b \in G)$$

(note that since L_a is diffeomorphism we can pushforward vector fields!)

Left invariant vector fields are completely determined by their values at e - identity element in G.

This enables to introduce additional structure in T_eG.

Take any vector $X_e \in T_eG$.

Define a left invariant vector field X by

$$X_a = L_{a*_e} X_e$$

Taking another $Y_e \in T_eG$ we have also Y s.t.

$$Y_a = L_{a*_e} X_e$$

We equipp T_eG with a structure of Lie algebra by setting

$$[X_e, Y_e] = [X, Y]_e .$$

Exercise: check that $L_{a*}[X,Y] = [L_{a*}X, L_{a*}Y]$

This is ok, since $L_{a*}[X,Y] = [L_{a*}X, L_{a*}Y] = [X,Y]$

How to define a left invariant metric on G?

Take <u>any</u> scalar product $\langle \, , \rangle_e$ in T_eG.

Define

$$\text{(LI)} \quad \boxed{g_a(X_a, Y_a) = \langle L_{a^{-1}*_a} X_a , L_{a^{-1}*_a} Y_a \rangle_e} \qquad \forall a\, G, \ \forall X_a \in T_aG$$

This is clearly left invariant because:

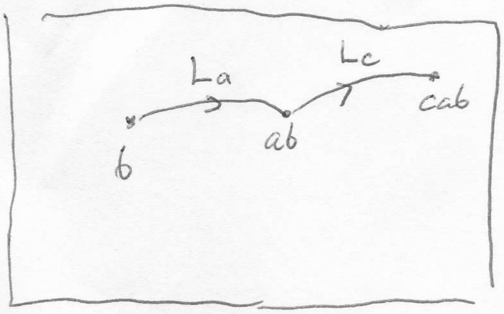
$$L_a \circ L_b = L_{ab} , \quad L_{c*_{ab}} \circ L_{a*_b} = L_{ca*_b}$$

In the same way we define right invariant metric on G.

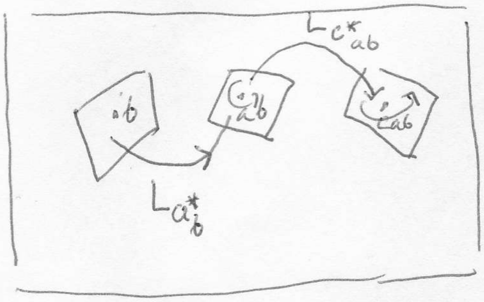
<u>Thm</u> see exercise 7 p. 46 in DoCarmo

Let G be a compact connected Lie group G.

Then G admits a bi-invariant Riemannian metric.



$$L_c \circ L_a = L_{ca}$$



$$L_{c^*_{ab}} \circ L_{a^*_b} = L_{ca^*_b}$$

$$g_{ab}\left(L_{a^*_b} X_b, L_{a^*_b} Y_b\right) =$$

$$= \langle L_{ab^{-1*}_{ab}} L_{a^*_b} X_b, L_{(ab)^{-1*}_{ab}} L_{a^*_b} Y_b \rangle_e =$$

$$= \langle L_{(b^{-1}a^{-1})^*_{ab}} L_{a^*_b} X_b, L_{(b^{-1}a^{-1})^*_{ab}} L_{a^*_b} Y_b \rangle_e =$$

$$= \langle L_{b^{-1*}_b} X_b, L_{b^{-1*}_b} Y_b \rangle_e = g_b(X_b, Y_b).$$

Example Upper half plane

$\mathbb{H}_+ = \{\mathbb{R}^2 \ni (x,y) : y > 0\}$

Group structure: on the space of functions $f_{(x,y)}: \mathbb{R} \to \mathbb{R}$

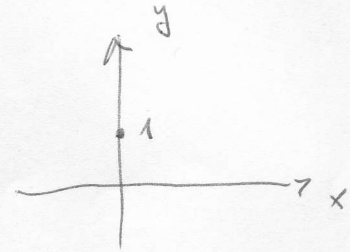
s.t. $f_{(x,y)}(t) = yt + x$ consider composition:

$$f_{(x',y')} \circ f_{(x,y)}$$

$$f_{(x',y')} \circ f_{(x,y)}(t) = f_{(x',y')}(f_{(x,y)}(t)) = f_{(x',y')}(yt+x) =$$

$$= y'yt + y'x + x' =$$

$$= f_{(y'x+x',\, y'y)}(t)$$



$$(x',y')\cdot(x,y) = (y'x + x', y'y) \in \mathbb{H}_+$$

Lie Group with $e = (0,1)$ and inverse $(x,y)^{-1} = \left(-\frac{x}{y}, \frac{1}{y}\right)$

Left invariant vector fields

Take ∂_x at e. $L_{(a,b)}(x,y) = (bx+a, by)$

$L_{(a,b)*}\partial_x|_e = b\,\partial_x|_{(a,b)}$ $\quad L_{(a,b)*} = \begin{pmatrix} b, & 0 \\ 0, & b \end{pmatrix}$

$X_1 = y\partial_x$ $\quad L_{(a,b)*}\partial_y|_e = b\,\partial_y|_{(a,b)}$

$X_2 = y\partial_y$

$[X_1, X_2] = -y\partial_x = -X_1$

Right invariant vector fields

$$R_{(a,b)}(x,y) = (x,y)\cdot(a,b) = (ya + x, yb)$$

$\partial_x \sim \gamma = (t,1)$; $R\gamma = (a+t, b) \Rightarrow R_*\partial_x = \partial_x$ $\quad Y_1 = \partial_x$

$\partial_y \sim (0, t+1) \Rightarrow R(0,t+1) = (a(t+1), b(t+1))$

$R_{(a,b)*}\partial_x|_e = \partial_x$ $\quad R_{(a,b)*}\partial_y|_e = a\partial_x + b\partial_y$ $\quad Y_2 = x\partial_x + y\partial_y$

$[Y_1, Y_2] = Y_1$

Left invariant metric which is δ_{ij} at $(0,1)$

$$\begin{cases} g(\partial_x,\partial_x) = \frac{1}{y}\cdot\frac{1}{y}\langle\partial_x,\partial_x\rangle_e = \frac{1}{y^2} \\ g(\partial_x,\partial_y) = \frac{1}{y}\cdot\frac{1}{y}\langle\partial_x,\partial_y\rangle_e = 0 \\ g(\partial_y,\partial_y) = \frac{1}{y}\cdot\frac{1}{y}\langle\partial_y,\partial_y\rangle_e = 0 \end{cases}$$

$$g = \frac{dx^2+dy^2}{y^2}$$

Right inv. metric which is δ_{ij} at $(0,1)$

$$g(\partial_x,\partial_x) = \langle\partial_x,\partial_x\rangle_e = 1$$

$$g(\partial_x,\partial_y) = \left\langle -\frac{x}{y}\partial_x + \frac{1}{y}\partial_y, \partial_x\right\rangle_e = -\frac{x}{y}$$

$$g(\partial_y,\partial_y) = \left\langle -\frac{x}{y}\partial_x + \frac{1}{y}\partial_y, -\frac{x}{y}\partial_x + \frac{1}{y}\partial_y\right\rangle = \frac{x^2}{y^2} + \frac{1}{y^2}$$

$$g = dx^2 - 2\frac{x}{y}dx\,dy + \frac{x^2+1}{y^2}dy^2 =$$

$$= \left(dx - \frac{x}{y}dy\right)^2 + \frac{1}{y^2}dy^2 =$$

$$= \frac{(y\,dx - x\,dy)^2 + dy^2}{y^2} = d\left(\frac{x}{y}\right)^2 + \left(\frac{dy}{y}\right)^2$$

$$\left(= dx'^2 + dy'^2 \qquad \begin{cases} x' = \frac{x}{y} \\ y' = \log y \end{cases} \right)$$

If g is a bi-invariant metric on G then the scalar product $\langle \, , \rangle$ induced by g in T_eG satisfies

$$\langle [U,X],Y \rangle + \langle X,[U,Y] \rangle = 0. \qquad (*)$$

And the oposite is true:

If we have a scalar product in T_eG such that $(*)$ holds then the metric defined by (LI) is biinvariant on G. (proof doCarmo p. 40-41)

4) Product metric

(M_1, g_1), (M_2, g_2)

Consider $M_1 \times M_2$ with projections $\pi_1 : M_1 \times M_2 \to M_1$
$\pi_2 : M_1 \times M_2 \to M_2$

$$g_1 \oplus g_2 (X,Y) := g_1(\pi_{1*}X, \pi_{1*}Y) + g_2(\pi_{2*}X, \pi_{2*}Y)$$

e.g. take a torus

$$\mathbb{T}^n = S^1 \times \dots \times S^1$$

and take a metric g on S^1 as an induced metric that S^1 gets from euclidean metric in $\mathbb{R}^2$.

$$\Rightarrow \quad g_{\mathbb{T}^n} = \underbrace{g_1 \oplus \dots \oplus g_1}_{n\text{-times.}}$$

flat torus

5) Every manifold (Hausdorff + countable basis) admits a Riemannian metric.

Partition of unity:

family of functions $f_\alpha : M \to \mathbb{R}$ s.t.

1) $\forall \alpha$ $f_\alpha \geq 0$ and $\text{supp}\, f_\alpha \subset U_\alpha$ — closure of the set of points where $f_\alpha \neq 0$.

2) $\{U_\alpha\}$ is a locally finite cover of M i.e.

$\bigcup_\alpha U_\alpha = M$, and $\forall p \in M$ $\exists W$ s.t $W \cap U_\alpha \neq \emptyset$ for only finite number of α — W: neighb. of p

3) $\sum_\alpha f_\alpha(p) = 1$ $\quad \forall p \in M$

Partition of unity always exists on M which is Hausdorff and has countable basis (see do Carmo p. 30)

$\Rightarrow$ take such a partition on M

$\{f_\alpha\}$, $\{U_\alpha\}$ coordinate charts

In each U_α we define a metric g^α s.t. in coordinate basis

$$g^\alpha\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = \delta^\alpha_{\mu\nu}$$

$$\Rightarrow \quad g = \sum_\alpha f_\alpha(p)\, g^\alpha.$$